Elementary maths for GMT

Calculus

Part 3.1: Multivariable calculus

Multivariable calculus

- *Multivariable calculus* is the branch of calculus that studies functions of more than one variable
- Multivariable generalizations of single-variable derivatives and integrals are *partial derivatives* and *multiple integrals*

Outline

1. Partial derivatives
2. Gradient
3. Directional derivative
4. Stationary points
5. Multiple integrals

Partial derivatives

<u>Definition</u>

The partial derivative $\frac{\partial f}{\partial x}$ of a function f to some variable x is defined as the derivative of f to x *while keeping all other variables fixed*, i.e. thought of as constants

- For example, take the following equation

$$f(x_1, x_2, x_3) = x_1{}^2 + \sin(x_2) + \cos(\ln(x_3))$$

  - The partial derivative with respect to x_1 is

$$\frac{\partial}{\partial x_1} f(x_1, x_2, x_3) = 2x_1$$

Examples

- With the function $f(x_1, x_2) = x_1 {x_2}^2$
  - regarding x_1 : $\dfrac{\partial}{\partial x_1} f(x_1, x_2) = {x_2}^2$
  - regarding x_2 : $\dfrac{\partial}{\partial x_2} f(x_1, x_2) = 2 x_1 x_2$

Examples

- With the function $f(x_1, x_2) = \cos(x_1 \sin(x_2))$
  - regarding x_1 :

$$\frac{\partial}{\partial x_1} f(x_1, x_2) = -\sin(x_1 \sin(x_2)) \sin(x_2)$$

  - regarding x_2:

$$\frac{\partial}{\partial x_2} f(x_1, x_2) = -x_1 \sin(x_1 \sin(x_2)) \cos(x_2)$$

Partial derivative shorthand

- Partial derivatives are often written in a more compact form using subscripts. For example:

$$\frac{\partial}{\partial x} f(x, y, z) = f_x$$

$$\frac{\partial}{\partial y} f(x, y, z) = f_y$$

etc.

Gradient

Definition

The gradient ∇f of a function f is a vector containing all the partial derivatives of f:

$$\nabla f(x_1, x_2, \dots, x_n) = \begin{pmatrix} f_{x_1} \\ f_{x_2} \\ \vdots \\ f_{x_n} \end{pmatrix}$$

Gradient operator

- The gradient operator ∇ is often called *nabla* or *del*. Instead of ∇f the notation *grad(f)* is also used
- The gradient operator can also be used ‘alone’ in formulas, where it stands for the vector of partial derivatives operator:

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}$$

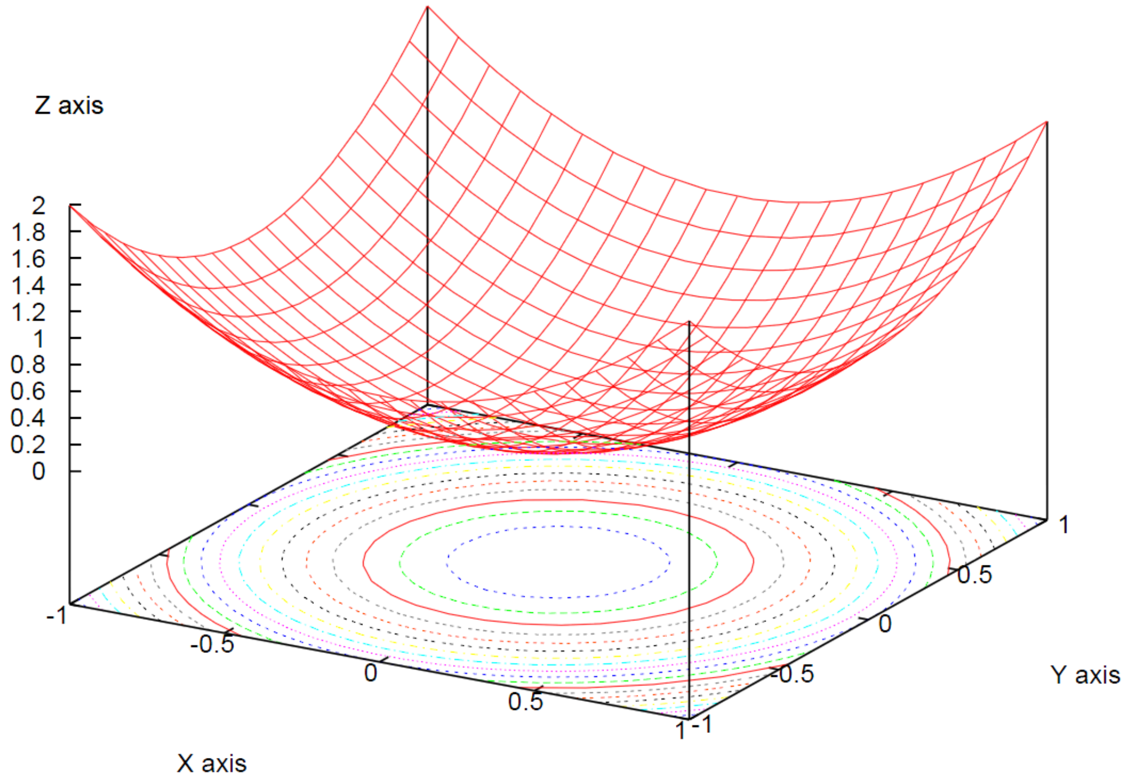
Gradient example and use

- The gradient at each point of a function is a vector in the direction of the locally steepest ascent

- Example

$$f(x_1, x_2) = {x_1}^2 + {x_2}^2$$

$$\nabla f(x_1, x_2) = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}$$



Directional derivative

<u>Definition</u>

Given a row vector $\boldsymbol{u}$, the *directional derivative* f_u of a function f in the direction of $\boldsymbol{u}$ is defined as

$$f_u = \frac{1}{\|\boldsymbol{u}\|}(\boldsymbol{u} \cdot \nabla f)$$

- This derivative equals the ‘normal’ (single-valued) derivative of the function *f* if you crossect it in the direction *u*
- Intuitively represents the instantaneous rate of change of *f* moving through a point in the direction ***u***

Illustration

- For three variables, let denotes $f = f(x, y, z)$ and $\boldsymbol{u} = (u, v, w)$, the definition turns into:

$$f_u = \frac{1}{\|\boldsymbol{u}\|}(uf_x + vf_y + wf_z)$$

- An alternative notation for f_u is $\nabla_u f$

Examples

$$f(x_1, x_2, x_3) = {x_1}^2 + {x_2}^2 + {x_3}^2 \text{ and } \boldsymbol{u} = (\frac{1}{2}, \frac{1}{2}\sqrt{3}, 0)$$

- The directional derivative of *f* in the direction of ***u*** at coordinates (1,2,3) is

$$\|u\| = \sqrt{(\frac{1}{2})^2 + (\frac{1}{2}\sqrt{3})^2 + 0} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$$f_u = \left(\frac{1}{2}, \frac{1}{2}\sqrt{3}, 0\right) \cdot \begin{pmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{pmatrix} = \frac{1}{2} \cdot 2x_1 + \frac{1}{2}\sqrt{3} \cdot 2x_2 + 0 \cdot 2x_3 = x_1 + \sqrt{3}x_2$$

$$f_u(1, 2, 3) = 1 + 2\sqrt{3}$$

Stationary points

Definition

A stationary point of a function of multiple variables is a point where all the partial derivatives are zero.

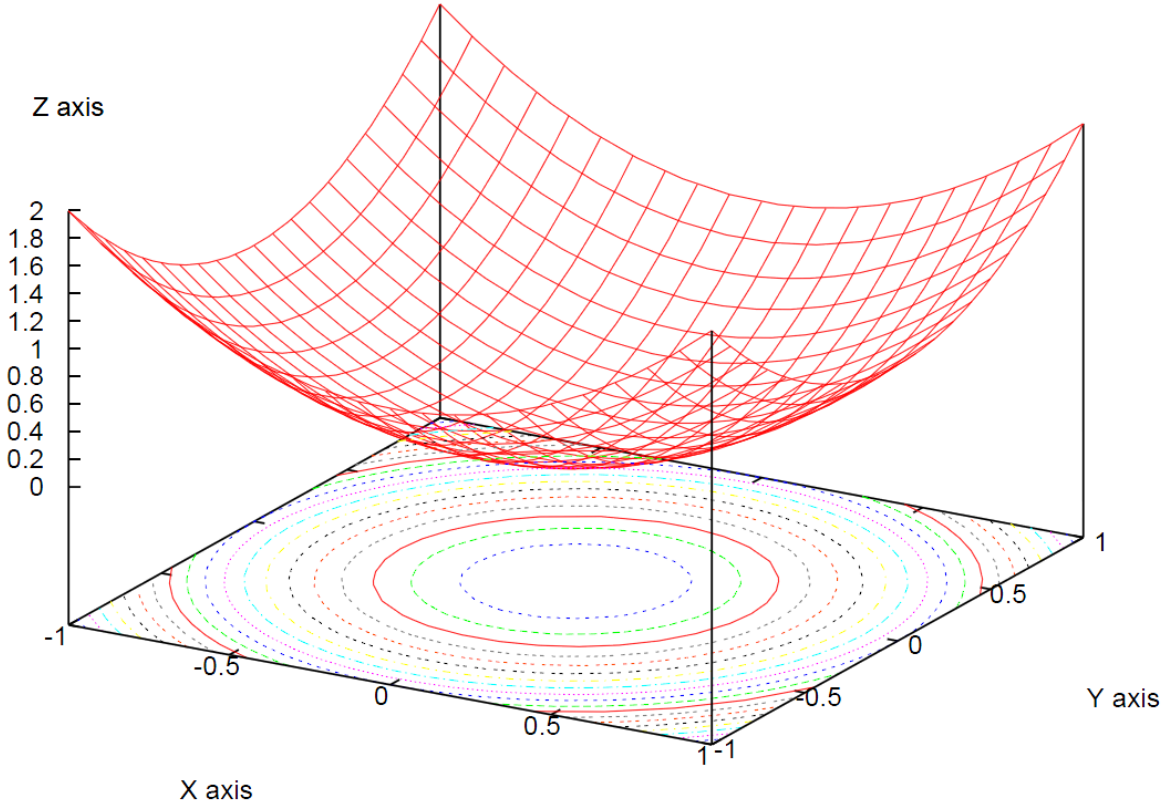
- Therefore, to find a stationary point the following should be solved

$$\begin{pmatrix} \dfrac{\partial}{\partial x_1} f(x_1, \dots, x_n) \\ \vdots \\ \dfrac{\partial}{\partial x_n} f(x_1, \dots, x_n) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Example

- Find the stationary point of the function

$$f(x, y) = x^2 + y^2$$



Example

- Find the stationary point of the function

$$f(x, y) = x^2 + y^2$$

$$\begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2x \\ 2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Thus the point (0,0) is the stationary point of the function *f(x,y)*

Type of stationary point

- There are two types of stationary points
  - extrema (minima or maxima)
  - saddle-points
- To determine the type, we need the *Hessian* of the function

Hessian

Definition

The Hessian H of a function $f(x_1, \dots, x_n)$ is the matrix of all second-order partial derivatives of the function f

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial {x_1}^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial {x_2}^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial {x_n}^2} \end{pmatrix}$$

Using the Hessian

- To find the type of a stationary point
  - The determinant $|H|$ of the Hessian of a function *f* evaluated in a stationary point *p* determines the type of stationary point
    - If $|H|$ > 0 and $\frac{\partial^2 f}{\partial {x_1}^2} < 0$ in stationary point p, then it is a maximum
    - If $|H|$ > 0 and $\frac{\partial^2 f}{\partial {x_1}^2} > 0$ in stationary point p, then it is a minimum
    - If $|H|$ < 0 in stationary point p, then it is a saddle-point

Example

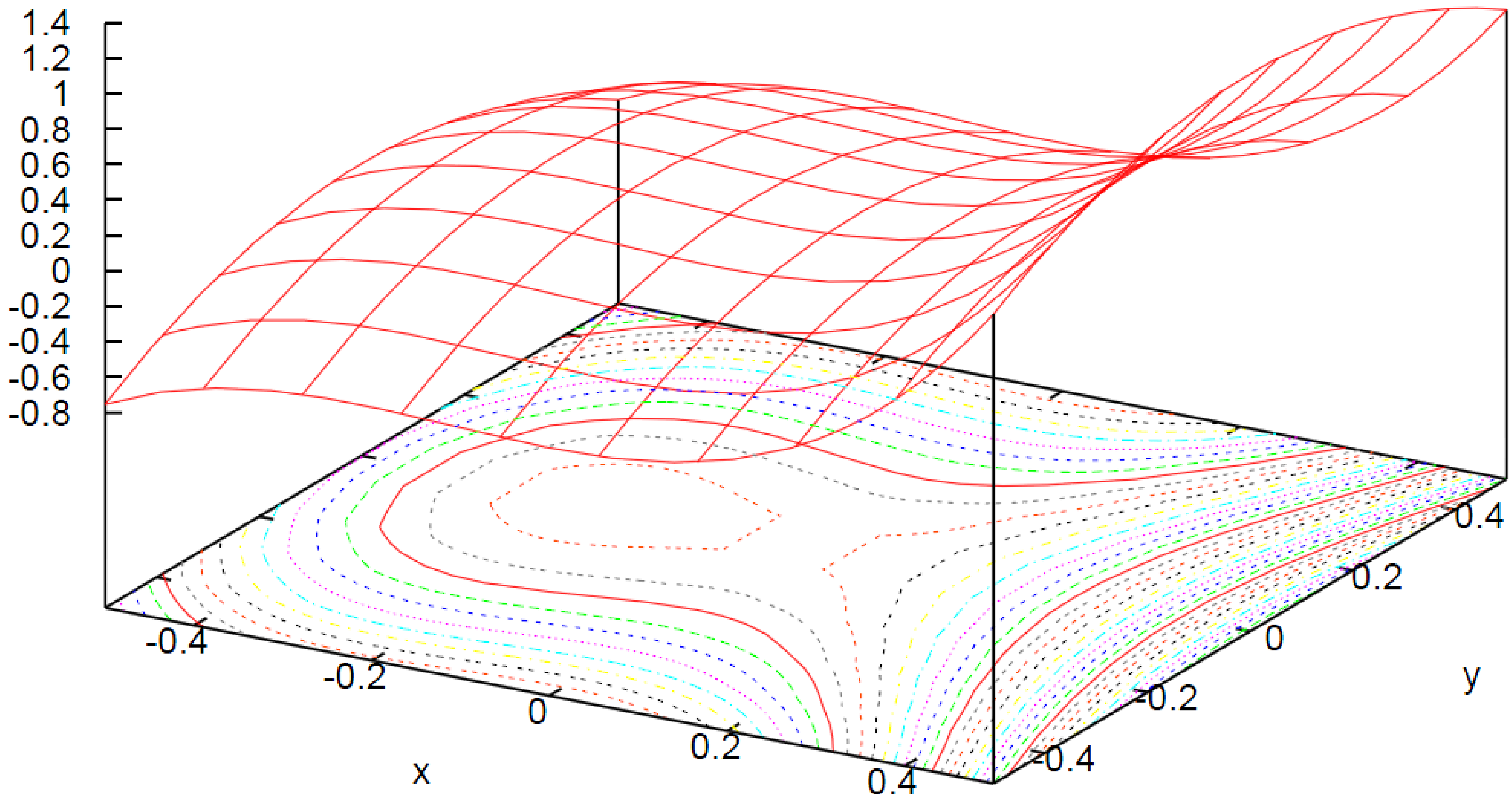
- Assuming the function $f(x, y) = 6x^3 + 2x^2 - 2y^2$
- Stationary points of *f* are (0,0) and $(-\frac{4}{18}, 0)$, and

$$|H| = \begin{vmatrix} 36x + 4 & 0 \\ 0 & -4 \end{vmatrix} = -144x - 16$$

  - At stationary point (0,0), |H| = -16
    Therefore (0,0) is a saddle-point
  - At stationary point $(-\frac{4}{18}, 0)$, |H| = 16, and $\frac{\partial^2 f}{\partial x^2} = -4 < 0$
    Therefore $(-\frac{4}{18}, 0)$ is a maximum

Example

$$f(x, y) = 6x^3 + 2x^2 - 2y^2$$



Multiple integrals

Definition

A multivariable function can in general be integrated to any of its variables. An integration to more than one variable is called a *multiple integral*.

- Example

$$\iint f(x, y)\, dy\, dx$$

  - This can often be computed by performing the integration from inside to outside:

$$\iint (f(x, y))dy\, dx = \int \left(\int (f(x, y))dy \right) dx$$

Meaning of multiple integrals

- In the same way that the definite integral of a function of a single variable corresponds to the area under the function, a 2D definite multiple integral corresponds to the *volume* under the function
- Single and multiple integrals are indispensible when computing areas and volumes of curved structures
  - They pop up in just about any advanced physics and mathematics problems, including many modeling, simulation and rendering problems

Example

- Integrate the following function *f* over the domain $(0,2) \times (1,3)$: $f(x,y) = x^2 + 2xy + y^2$

$$\int_0^2 \int_1^3 (x^2 + 2xy + y^2) dy\, dx = \int_0^2 \left[x^2 y + xy^2 + \frac{1}{3} y^3\right]_{y=1}^{y=3} dx$$

$$= \int_0^2 (3x^2 + 9x + 9 - (x^2 + x + \frac{1}{3}))\, dx$$

$$= \int_0^2 2x^2 + 8x + \frac{26}{3}\, dx$$

$$= \left[\frac{2}{3} x^3 + 4x^2 + \frac{26}{3} x\right]_0^2$$

$$= \frac{16}{3} + 16 + \frac{52}{3} - (0 + 0 + 0) = \frac{116}{3}$$